Journal of Engineering Physics and Thermophysics, Vol. 71, No. 2, 1998

# ANALYTICAL AND NUMERICAL SOLUTION OF THE PROBLEM OF METEOROID MOTION IN TERRESTRIAL ATMOSPHERE

N. A. Kudryashov and E. A. Mukhai

UDC 523.68

*A system of equations describing meteoroid motion in the terrestrial atmosphere is considered. It is shown that the system can be reduced to a single second-order differential equation for the height dependence of the mass. Approximate analytical expressions for the solution of the Cauchy problem for this equation are obtained, and conditions of applicability of these solutions are determined. The general case of the problem is solved numerically. Results of mathematical modeling are presented.*

**Introduction.** Meteoroids entering the terrestrial atmosphere interact with atoms and molecules of air. The entire meteoroid flight can be divided provisionally into two portions, where its behavior is determined by different values of the Knudsen number. These two portions of the flight correspond to large and small Knudsen numbers ($Kn \geq 0.1$ and $Kn \leq 0.1$ [1]. The main difference in the meteoroid flight within these two regions is connected with the effect of atoms and molecules of air. At large Knudsen numbers, this effect is insignificant and does not lead to a noticeable change in the meteoroid mass during the flight. In the second portion of the flight, considerable changes in the meteoroid mass take place, which affects substantially the change in parameters of the moving meteoroid.

The change of the meteoroid mass in the course of flight, known as ablation, is explained by the effect of several factors [2]. At heights exceeding 120 km, collisions with separate atoms and molecules of air rarely occur. At heights lower than 120 km, when the terrestrial atmosphere becomes more dense, the effect of blodcing of the meteoroid by ablated molecules takes place. At heights of 1–2 km, when the terrestrial atmosphere becomes substantially more dense, intense evaporation of the meteoroid material takes place, which is connected with the rapid heating of its surface as a result of collisions with molecules of air.

The kinetic energy being lost by the moving meteoroid is distributed over different portions of its flight differently. Prior to an intense evaporation of the meteoroid material, approximately half the kinetic energy being lost is carried away by recoiled air molecules, and the rest is carried away by the material being evaporated. At lower flight heights when the evaporation increases, the main share of the kinetic energy is carried away by evaporated molecules. In this case, energy losses are even more complicated due to the uncoupled shock wave being formed.

Another mechanism of mass loss can take place in the course of meteoroid motion, which is connected with melting of the outer layer and removal of the melt film formed by the approaching air stream. It has been shown in a number of investigations that stony meteoroids lose a substantial share of their mass due to the melting and removal process [2], and iron meteoroids in this case lose almost the entire mass.

In addition to the above-enumerated factors leading to a change in the mass of the moving meteoroid, data can be found which bear witness to the fact that most meteoroids break down into several pieces in the course of their flight. In a number of cases, moving meteoroids undergo, in essence, continuous fragmentation.

It follows from the above considerations that meteoroid ablation is a rather complicated process. This circumstance leads to uncertainties in estimates of parameters of the problem of the meteoroid motion. In the present work, we attempt to find an analytical solution of the problem of meteoroid motion and compare it with results of the mathematical modeling of this phenomenon.

---

Moscow Engineering Physics Institute, Moscow, Russia. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 71, No. 2, pp. 342-348, March-April, 1998. Original article submitted June 30, 1996.

 

In what follows, we assume that the parameters of the problem of meteoroid motion change within a range of values, and the meteoroid motion is studied at different values of these parameters. Comparison of results of a numerical experiment with experimental data will make it possible to adjust the values of the parameters and reveal the contribution of each of the factors to meteoroid flight.

**Formulation of the Problem and Analytical Solutions.** The system of equations of the physical theory of meteoroid motion written under the assumption of motion of the body within a plane including the axis $z$ is as follows [1]:

$$m\frac{dv}{dt} = mg\sin\theta - \frac{1}{2}\rho v^2 c_1 S - fv\frac{dm}{dt}, \tag{1}$$

$$\frac{dz}{dt} = -v\sin\theta, \tag{2}$$

$$mv\frac{d\theta}{dt} = -\frac{mv^2\cos\theta}{z+R_0} + mg\cos\theta - c_2\frac{S}{2}\rho v^2, \tag{3}$$

$$e_0\frac{dm}{dt} = -\frac{c_3}{2}S\rho v^3. \tag{4}$$

It is assumed in Eqs. (2) and (3) that the change in the density with the height obeys the barometric formula

$$\rho(z) = \rho_0\exp\left(-\frac{z}{H}\right). \tag{5}$$

The reactive recoil coefficient $f$ changes from $-1$ to 1. The coefficient $c_1$ decreases with the height and changes within the limits of 0.9 to 2 with variations in the Knudsen number Kn from 0.1 to 10.

The coefficient $c_3$ consists of the convective and radiative portions of the heat transfer. The coefficient depends on the velocity $v$, air density $\rho$, and the average meteoroid size $r_0$. In the case of large bodies, the radiative heat transfer plays the main part, and, for example, as has been shown by computations for a meteoroid with a radius of 0.5 m, $c_3$ varies within the limits of 0.01 to 0.1. The coefficient of the lifting power $c_2$ is small, and its effect on the meteoroid motion is usually not taken into account [3].

Another important parameter of the moving meteoroid being observed is the luminosity intensity, calculated by the formula

$$I = -\frac{1}{2}\tau_0 v^2\frac{dm}{dt} = \frac{\tau_0 c_3\rho_0 S}{4e_0}v^5\exp\left(-\frac{z}{H}\right). \tag{6}$$

Here $\tau_0$ is an experimentally evaluated coefficient. For the range of meteoroid velocities of 10 to 80 km/sec, the value of $\tau_0$ lies in the range of $0.6\cdot 10^{-3}$ to $1.0\cdot 10^{-3}$ [3].

The system of equations (1)-(4) is closed by the obvious relationships

$$m = \frac{4}{3}\pi\rho_m r_0^3 A, \quad S = \pi r_0^2. \tag{7}$$

The system of equations (1)-(7) is solved under the assumption that in the initial instant the coordinates, velocity, and mass of the meteoroid are given: $z(t=0) = z_0$, $\theta(t=0) = \theta_0$, $v(t=0) = v_0$, $m(t=0) = m_0$.

The solution yields the parameters of the moving meteoroid $z(t)$, $\theta(t)$, $v(t)$, $m(t)$, and $I(t)$, which depend parametrically on the coefficients $c_1$, $c_2$, $c_3$, and $f$ discussed above.

In calculations of the meteroid motion, the height at which the force of the terrestrial attraction is equilibrated by the resistance force is frequently taken as the initial coordinate $z_0$. Without accounting for the mass change, one obtains from Eq. (1)

$$m_0 g \sin\theta_0 = \frac{1}{2}\rho_0 v_0^2 c_1 S \exp\left(-\frac{z_0}{H}\right), \tag{8}$$

from which $z_0$ can be found at given $m_0$, $v_0$, and $\theta_0$:

$$z_0 = H \ln\left(\frac{\rho_0 v_0^2 c_1 S}{2 m_0 g \sin\theta_0}\right). \tag{9}$$

Equation (1), not taking into account the mass change and written for the fixed angle $\theta = \theta_0$, can be i..tegrated, and one can obtain the $v(z)$ dependence.

By dividing Eq. (1) by Eq. (2), we obtain

$$\frac{dv}{dz} = -\frac{g}{v} + \frac{c_1 S \rho_0}{2m\sin\theta} v \exp\left(-\frac{z}{H}\right), \tag{10}$$

from which, after the substitution $y = v^2/2$, $\chi_0 = c_1\rho_0 S/(m\sin\theta)$, we arrive at the equation

$$\frac{dy}{dz} = -g + \chi_0 \exp\left(-\frac{z}{H}\right) y, \tag{11}$$

which upon integration by the Lagrange-multiplier method yields the dependence of the meteoroid velocity on the coordinate:

$$v(z) = \sqrt{2}\left\{\frac{1}{2} v_0^2 \exp\left[\chi_0 H \exp\left(-\frac{z_0}{H}\right)\right] - gH \int_{\varphi(z)}^{\varphi(z_0)} \frac{\exp(-x)}{x} dx\right\}^{1/2} \times \exp\left[-\frac{\chi_0 H}{2}\exp\left(-\frac{z}{H}\right)\right], \tag{12}$$

where

$$\varphi(x) = -\chi_0 H \exp\left(-\frac{x}{H}\right).$$

For $z$ close to $z_0$ determined by Eq. (9), one can arrive, with regard for Eq. (12), at the celebrated asymptotic expression

$$v = v_0 \exp\left[-B\left(\exp\left(-\frac{z}{H}\right) - \exp\left(-\frac{z_0}{H}\right)\right)\right], \tag{13}$$

where $B = \rho_0 g H c_1 S/(2mg\sin\theta)$ is the ballistic coefficient.

Now let us consider the solution of the problem of meteroid motion at large Knudsen numbers, taking into account disintegration of the meteoroid.

With regard for (4), Eq. (1) can be written as follows:

$$m\frac{dv}{dt} = mg\sin\theta - \frac{1}{2}\rho v^2 c_1 S + \frac{c_3 f \rho}{2e_0} S v^4. \tag{14}$$

By dividing this equation by Eq. (2), we obtain

$$m\frac{dv}{dz} + \frac{mg}{v} - \frac{c_1\rho_0}{2\sin\theta} S v \exp\left(-\frac{z}{H}\right) + \frac{c_3 f\rho_0}{2e_0\sin\theta} S v^3 \exp\left(-\frac{z}{H}\right) = 0. \tag{15}$$

Upon multiplying by $v$, this equation takes the form

$$\frac{m}{2}\frac{dv^2}{dz} + mg - \frac{c_1 \rho_0}{2\sin\theta} Sv^2 \exp\left(-\frac{z}{H}\right) + \frac{c_3 f \rho_0}{2e_0 \sin\theta} Sv^4 \exp\left(-\frac{z}{H}\right) = 0\,. \tag{16}$$

With regard for (2) and (4), we obtain

$$v^2 = \frac{2e_0 \sin\theta}{c_3 \rho_0 S} \exp\left(\frac{z}{H}\right) \frac{dm}{dz}\,. \tag{17}$$

Since

$$S = \frac{\pi 3^{2/3}}{(4\pi \rho_m A)^{2/3}} m^{2/3}\,, \tag{18}$$

we have

$$v^2 = B_0\, m^{-2/3} \frac{dm}{dz} \exp\left(\frac{z}{H}\right), \quad B_0 = \frac{2e_0 \sin\theta\,(4\pi \rho_m A)^{2/3}}{c_3 \rho_0\, \pi 3^{2/3}}\,. \tag{19}$$

By substituting (18) and (19) into Eq. (16), we arrive at the equation describing the change in meteoroid mass in the course of its motion:

$$m\frac{d^2 m}{dz^2} + 2\left(f - \frac{1}{3}\right)\left(\frac{dm}{dz}\right)^2 + \frac{1}{H} m \frac{dm}{dz} + \frac{2g}{B_0} \exp\left(-\frac{z}{H}\right)\left(m - \frac{c_1 e_0}{c_3 g}\frac{dm}{dz}\right) m^{2/3} = 0\,. \tag{20}$$

For Eq. (20), at a given initial meteoroid mass $m(t=0) = m_0$ and velocity $v(t=0) = v_0$, we have the Cauchy problem with

$$m\,(z = z_0) = m_0 \tag{21}$$

and

$$\frac{dm}{dz}(z = z_0) = \frac{v_0^2 m_0^{2/3}}{B_0} \exp\left(-\frac{z_0}{H}\right). \tag{22}$$

By solving problem (20)-(22), we evaluate $m(z)$, and then, by formula (11), $v(t)$. The time dependence of the $z$ coordinate can be found by solving Eq. (2) with the given initial height $z(t=0) = t_0$.

Let us introduce dimensionless variables and parameters by the following formulas:

$$m' = \frac{m}{m_0}, \quad z' = \frac{z}{H}, \quad v' = \frac{v}{v_0}, \quad \chi_1 = \frac{2gH^2}{B_0 m_0^{1/3}}, \quad \alpha_1 = \frac{c_1 e_0}{c_3\, gH}\,. \tag{23}$$

Then, Eqs. (19) and (20) will assume the following form (in what follows, we omit primes):

$$v\,(z) = \frac{1}{v_0}\left(\frac{B_0 m_0^{1/3}}{H}\right)^{1/2}\left(\frac{dm}{dz}\exp\,(z)\, m^{-2/3}\right)^{1/2}, \tag{24}$$

$$mm_{zz} + 2\left(f - \frac{1}{3}\right) m_z^2 + mm_z + \chi_1 \exp\,(-z)\,(m - \alpha_1 m_z)\, m^{2/3} = 0\,, \tag{25}$$

$$m\ (z = z_0) = 1\ , \tag{26}$$

$$\frac{dm}{dz}(z = z_0) = \frac{v_0^2 H}{B_0 m_0^{1/3}} \exp\left(-\frac{z_0}{H}\right). \tag{27}$$

The system of equations (24), (25) with conditions (26), (27) can be conveniently used in the mathematical modeling of meteroid motion in the terrestrial atmosphere.

An analytical solution of the Cauchy problem (25)-(27) cannot generally be found. Therefore, we will seek the solution under the assumption that the force of gravity in Eq. (25) is compensated by the drag force. Equation (25) in this case transforms into

$$mm_{zz} + 2\left(f - \frac{1}{3}\right) m_z^2 + mm_z = 0\,. \tag{28}$$

It can be solved by the substitution $m(z) = \exp\ [\int \tilde{m}(z)dz]$, and the solution is expressed, with regard for (26) and (27), as follows:

$$m\ (z) = \begin{cases} \left[1 - \dfrac{2v_0^2 H}{B_0 m_0^{1/3}}\left(f + \dfrac{1}{6}\right)(\exp(-z) - \exp(-z_0))\right]^{\frac{1}{2(f+1/6)}}, & f \neq -\dfrac{1}{6}; \\ \exp\left[-\dfrac{v_0^2 H}{B_0 m_0^{1/3}}(\exp(-z) - \exp(-z_0))\right], & f = -\dfrac{1}{6}. \end{cases} \tag{29}$$

It follows from (29) that when $f > -1/6$, the meteoroid mass approaches zero at the height

$$z_1 = z_0 - \ln\left[1 + \frac{B_0 m_0^{1/3}}{2v_0^2 H\left(f + \frac{1}{6}\right)} \exp(z_0)\right] \tag{30}$$

from which follows the relationship

$$m_0 = \left[\frac{2v_0^2 H\left(f + \frac{1}{6}\right)}{B_0}(\exp(-z_1) - \exp(-z_0))\right]^3, \tag{31}$$

which makes it possible to evaluate the initial meteoroid mass if the height at which it burns away completely is known.

By substituting (29) into (24), we obtain the change of the meteoroid velocity with height:

$$v\ (z) = \begin{cases} \left[1 - \dfrac{2v_0^2 H}{B_0 m_0^{1/3}}\left(f + \dfrac{1}{6}\right)(\exp(-z) - \exp(-z_0))\right]^{-\frac{f}{2(f+1/6)}}, & f \neq -\dfrac{1}{6}; \\ \exp\left[-\dfrac{v_0^2 H}{6B_0 m_0^{1/3}}(\exp(-z) - \exp(-z_0))\right], & f = -\dfrac{1}{6}. \end{cases} \tag{32}$$

By taking account of (29) and (32), we obtain from Eq. (6) the dependence of the meteoroid luminosity on height. In dimensioned variables it can be expressed as follows:

$$I = \frac{\tau_0 c_3 \rho_0 (\pi 3^{2/3}) m_0^{2/3} v_0^5}{4 e_0 (4\pi \rho_m A)^{2/3}} \exp\left(-\frac{z}{H}\right) \times$$

$$\times \begin{cases} \left[1 - \dfrac{2v_0^2 H}{B_0 m_0^{1/3}}\left(f + \dfrac{1}{6}\right)\left(\exp\left(-\dfrac{z}{H}\right) - \exp\left(-\dfrac{z_0}{H}\right)\right)\right]^{-\frac{15f-2}{6f+1}}, & f \neq -\dfrac{1}{6}; \\ \exp\left[-\dfrac{3v_0^2 H}{2B_0 m_0^{1/3}}\left(\exp\left(-\dfrac{z}{H}\right) - \exp\left(-\dfrac{z_0}{H}\right)\right)\right], & f = -\dfrac{1}{6}. \end{cases} \tag{33}$$

It should be noted that, in the approximation being considered, the following relationship holds:

$$v\, m^f = \text{const}\,, \tag{34}$$

which is the integral of motion of the moving meteoroid with regard for its disintegration.

Finally, we consider another widespread model of meteoroid motion accounting for its slowing down and disintegration [3]. It coincides with problem (25)-(27), where the effect of the force of gravity is not accounted for and it is assumed that $f = 0$. In this case, we obtain from Eq. (16)

$$\frac{m}{2}\frac{dv^2}{dz} + mg - \frac{c_1 e_0}{c_3}\frac{dm}{dz} = 0\,, \tag{35}$$

from which we obtain the energy integral for the moving meteoroid with regard for its disintegration:

$$\frac{v^2}{2} + gz - \left(\frac{c_1 e_0}{c_3}\right)\ln\frac{m}{m_0} = b_1\,. \tag{36}$$

With characteristic values of parameters of the problem, we have $gz \ll v^2$. Therefore, the term $gz$ can be neglected and, with regard for the initial data, the constant takes the form $b_1 = v_0^2/2$.

By substituting (19) into (35), we arrive at the equation of the first kind with respect to $m(z)$:

$$\frac{B_0}{2} m^{-2/3}\frac{dm}{dz}\exp\left(\frac{z}{H}\right) + gz - \left(\frac{c_1 e_0}{c_3}\right)\ln\frac{m}{m_0} = \frac{v_0^2}{2}\,. \tag{37}$$

Without taking into account the effect of the force of gravity in Eq. (37), its solution with regard for the condition $m(z = 0) = m_0$ is as follows:

$$z = z_0 - H\ln\left[1 - \frac{c_3 B_0 m_0^{1/3}}{2c_1 H e_0}\exp\left(\frac{z_0}{H} - \alpha\right)\int_{x_1(m)}^{x_2(m)}\frac{\exp(-x)}{x}\,dx\right]. \tag{38}$$

Here

$$\alpha = \frac{c_3 v_0^2}{6c_1 e_0}\,; \quad x_1 = -\alpha\,; \quad x_2 = -\alpha\left[1 + \frac{1}{3\alpha}\ln\left(\frac{m}{m_0}\right)\right].$$

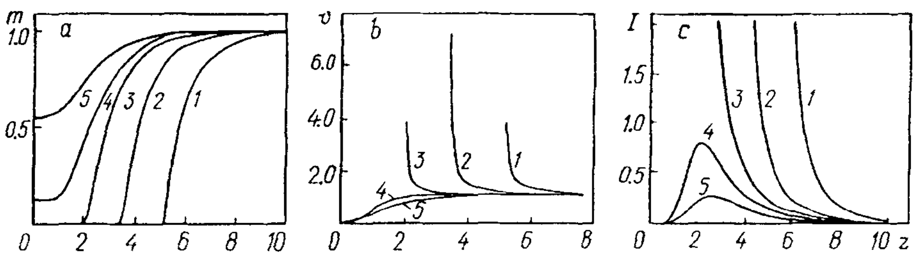


Fig. 1. Dependences of mass (a), velocity (b), and luminosity (c) of a moving meteoroid on (dimensionless) height $z$ at $c_1 = 0.9$ and $f = 1$ and different values of the parameter $k = e_0/c_3$: 1) $k_1 = 10$; 2) $k_2 = 50$; 3) $k_3 = 150$; 4) $k_4 = 250$; 5) $k_5 = 500$ MJ/kg.

**Results of Mathematical Modeling.** Generally, as has been noted above, an analytical solution of the problem of meteoroid motion cannot be found. Therefore, to find the solution of the problem within a wide range of variation of parameters, we used a numerical method. The Cauchy problem (25)-(27) was solved numerically by the fourth-order Runge–Kutta method [4].

Figure 1 illustrates results of calculations of the characteristics of meteoroid motion at various parameters of the problem. It is evident that at a certain value of $k$, the character of the dependences changes. This takes place at $k \cong 220$ MJ/kg. When $k < 220$ MJ/kg, the meteoroid disintegrates completely in the terrestrial atmosphere. When $k > 220$ MJ/kg, it reaches the surface of the Earth. Disintegration of the meteoroid in the atmosphere depends on its composition and the coefficient of the heat transfer to the ambient air. It is usually assumed that the maximum value of $e_0$ is 5000 kJ/kg and the minimum value of the coefficient $c_3 = 0.01$, which leads to the equation $k = k_5$. Therefore, at the maximum disintegration enthalpy of the meteoroid and minimum coefficient of heat transfer, it follows from Fig. 1a that the meteoroid mass decreases approximately twofold.

Analytical solutions $m(z)$, $v(z)$, and $I(z)$ for the motion of a meteoroid presented by Eqs. (29), (32), and (33) at values of $f = 1$ and $c_1 = 0.9$ are described well by curve 1 and satisfactorily by curve 2, which means that the analytical solution can be used to describe meteoroid motion when $k < k_z = 50$ MJ/kg.

This work was carried out with the support of the ISTC project, grant No. V23-96.

## NOTATION

$m$, meteoroid mass; $v$, meteoroid velocity; $z$, vertical coordinate; $\theta$, angle of inclination with respect to the horizon; $t$, time; Kn, Knudsen number; $g$, free-fall acceleration; $f$, coefficient of reactive force; $c_1$, air-drag coefficient; $c_2$, lifting-force coefficient; $c_3$, heat-transfer coefficient; $e_0$, effective disintegration enthalpy; $\tau_0$, coefficient of luminosity efficiency; $S$, area of meteoroid surface; $A$, form-factor ($A = 1$ for sphere); $R_0$, radius of the Earth; $B$, ballistic coefficient; $\rho$, atmosphere density; $\rho_0$, air density at sea level ($\rho_0 = 1.29$ kg/m$^3$); $\rho_m$, density of meteoroid substance ($\rho_m = 8000$ kg/m$^3$); $H$, height of homogeneous atmosphere ($H \cong 7000$ m); $l_0$, size of meteoroid.

## REFERENCES


1. V. P. Korobeinikov, Znanie, Ser. Matemat. Kibernet., No. 1, 24-28 (1986).
2. V. A. Bronshtein, in: Meteors, Meteorites, and Meteoroids [in Russian], Moscow (1987), pp. 39-89.
3. I. V. Nemchinov, O. P. Popova, V. V. Svetsov, and V. V. Shuvalov, Astron. Vestn., **29**, No. 2, 155-173 (1995).
4. N. N. Kalitkin, Numerical Methods [in Russian], Moscow (1978).